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NECESSARY AND SUFFICIENT CONDITIONS FOR DYNAMIC OPTIMIZATION

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We characterize the necessary and sufficient conditions for optimality in discrete-time, infinite-horizon optimization problems with a state space of finite or infinite dimension. It is well known that the challenging task in this problem is to prove the necessity of the transversality condition. To do this, we follow a duality approach in an abstract linear space. Our proof resembles that of Kamihigashi (2003), but does not explicitly use results from real analysis. As an application, we formalize Sims's argument that the no-Ponzi constraint on the government budget follows from the necessity of the tranversality condition for optimal consumption.

Keywords: Dynamic Optimization, Transversality Condition

1. INTRODUCTION

We characterize the necessary and sufficient conditions for optimality in discrete-time, infinite-horizon optimization problems with a state space of finite or infinite dimension. It is well known that the challenging task in this problem is to prove the necessity of the transversality condition (TVC)—see Stokey and Lucas (1989). To do this, we follow a duality approach and construct support prices using a separating hyperplane theorem. This approach highlights the economic intuition behind the optimality conditions and lends itself to useful applications. We demonstrate such a case at the end of the paper, where we derive the no-Ponzi condition for a fiscal dynamic optimization problem from the budget constraint and TVC of households.

Our work contributes to an established literature in optimization. Weitzman (1973) characterizes optimality conditions in a discrete-time infinite-horizon setting by restricting the state space to be Euclidean. Benveniste and Scheinkman

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(1982) provide a continuous-time treatment. Michel (1990) extends the characterization in Weitzman (1973) to a general concave discrete-time optimal control problem. Kamihigashi (2001) generalizes the necessity result for the TVC without assuming concavity.

These studies, however, mostly focus on finite-dimensional state spaces. Infinite-dimensional state spaces arise in economics. For example, in a stochastic economy, the state is a function of random shocks, and optimization theory can be extended straightforwardly to such a stochastic environment by representing the state at a given time as a random vector. In general, such random vectors are elements of an infinite-dimensional vector space. Although the characterization of Michel (1990) can be generalized by assuming a nonempty relative interior for the infinite-dimensional set, Kamihigashi (2003) proves the necessity of the TVC directly in a stochastic dynamic problem, utilizing some results (such as Fatou's lemma) that are specific to vector spaces of random variables.

Our modest contribution is to characterize all of the optimality conditions for an infinite-dimensional state space in general vector-space terms, in a framework chosen judiciously to be highly tractable but not unduly narrow. By doing so, we provide a geometrically intuitive statement of the conditions that unify the finite- and infinite-dimensional cases. For the sake of intuitiveness and in order to keep proofs simple, we have stated some conditions in less generality than would be possible. In a concluding section, we indicate how to handle an issue regarding the dual space (that is, the space of price vectors) by relaxing one of our assumptions.

Before entering a technical discussion, let us briefly summarize the intuition behind the theory to be studied. This theory concerns optimization over an infinite (without loss of generality) time horizon. The optimization problem is to control the evolution of a *state,* such as a vector of stocks of various capital goods. There is an exogenous *initial state*. At each time (as represented hereafter by the discrete sequence of dates $1, 2, \ldots$), there is an exogenous constraint on which new states can be reached from the current state. For example, what capital stock would it be feasible to hold next year, given the amount of capital currently held? A choice must be made, subject to this constraint, and a momentary payoff results from the choice. For example, making a transition to a relatively low capital stock at the next state leaves more resources available for consumption than reaching a higher stock would allow, so choosing the lower stock will afford a higher momentary payoff. But the higher stock at the next date would afford more opportunities and, presumably, a higher momentary payoff at that next date. Thus, the optimization problem is to achieve the best stream of payoffs, according to some intertemporal aggregation criterion.

2. NOTATION AND DEFINITIONS

The state x_t is a vector defined over a Banach space V (that is, a vector space with a norm that induces a complete topology) with the norm $\|.\|_v : V \to \mathbf{R}_+$.[1] Suppose that V is partially ordered by the binary relation $\leq_V$ such that for all x,

y, and z in V and $\alpha > 0$ in $\mathbf{R}$, the following relation holds:

$$x \leq_V y \Rightarrow [x + z \leq_V y + z \text{ and } \alpha x \leq_V \alpha y].$$

Define the positive cone of V by $P = \{x_t \in V \mid 0 \leq_V x_t\}$. The normed-dual space V^* of V is the vector space of continuous linear functionals $p : V \to \mathbf{R}$, with the positive cone $P^* = \{p \in V^* \mid \forall x_t \in P \;\; 0 \leq p(x_t)\}$.[2]

The product space $V \times \mathbf{R}$ is of special interest in the context of dynamic programming where the state vector and its value are key objects. Because normed vector spaces are closed under Cartesian products with a suitably chosen norm, the product space $V \times \mathbf{R}$ is also a normed vector space.

Transforming the state variable x_t to x_{t+1} yields a payoff of u_t. Preferences and technology are thus captured by the set

$$(x_t, u_t, x_{t+1}) \in \Phi_t \subseteq V \times \big(\mathbf{R} \cup \{-\infty\}\big) \times V.$$

In contrast to Weitzman (1973), we do not restrict momentary payoffs to be bounded from below. Such a restriction would be inconsistent with some formulations that are convenient in applications, especially with specifying a logarithmic payoff function. The need to cope with the possibility of unbounded payoffs will motivate our choice of an intertemporal-payoff-aggregation criterion in the ensuing analysis.

At each t, the set of states that allow a technologically feasible transition is denoted by

$$X_t = \{x \;\big|\; \exists u \; \exists x' \; (x, u, x') \in \Phi_t\}.$$

A *path* is a sequence $\langle x_t, u_t \rangle_{t=0}^{\infty} \in V^{\mathbf{N}} \times (\mathbf{R} \cup \{-\infty\})^{\mathbf{N}}$, where $\langle x_\tau, u_\tau \rangle_{\tau=t}^{\infty}$ denotes the infinite sequence $\langle (x_0, u_0), (x_1, u_1), (x_2, u_2), \ldots \rangle$. Each ordered pair specifies a state and a payoff level. The set of *feasible paths* from date t onward, starting with an initial state $x_t = \overline{x}$, is defined as

$$F_{t,\overline{x}} = \left\{ \langle x_\tau, u_\tau \rangle_{\tau=t}^{\infty} \mid x_t = \overline{x} \text{ and for } \tau \geq t,\, (x_\tau, u_\tau, x_{\tau+1}) \in \Phi_\tau \right\}.$$

We make the following assumptions about the primitives of the problem. Additional assumptions that put more structure on the relevant objects will be introduced later.

A.1. *If $\overline{x} \in X_t$, then $F_{t,\overline{x}} \neq \emptyset$.*[3]

A.2. *For all t, given some $\overline{x} \in X_t$, the following condition holds for all $\langle x_\tau, u_\tau \rangle_{\tau=t}^{\infty} \in F_{t,\overline{x}}$:*

$$\liminf_{T \to \infty} \sum_{\tau=t}^{T} u_\tau < \infty.$$

(A.1) states that there exists a feasible path. In a multisector growth model with an unbounded technology set, the upper bound in (A.2) would follow from an underlying scarcity of labor, as in Peleg and Ryder (1972), and McKenzie (1976) obtains it also from a set of assumptions about the underlying momentary utility function being concave and the optimal state being interior. We prefer to introduce it directly.

Because infinite payoff sums may not converge, we use the catching-up criterion for optimality introduced by Gale (1967), which compares finite partial sums. A path $\langle x_t, u_t\rangle_{t=0}^{\infty}$ is said to *catch up* with $\langle y_y, v_t\rangle_{t=0}^{\infty}$ if

$$\liminf_{T\to\infty} \sum_{t=0}^{T} (u_t - v_t) \geq 0. \tag{1}$$

An *optimal* path catches up with all other feasible paths having the same initial condition. A path $\langle x_t, u_t\rangle_{t=0}^{\infty} \in F_{0,\overline{x}}$ is *optimal* given initial state $\overline{x}$ if and only if it catches up with all $\langle y_t, v_t\rangle_{t=0}^{\infty} \in F_{0,y_0}$, where $y_0 \leq \overline{x}$ and $y_0 \in X_0$. Note that for convergent sums, this definition is equivalent to the definition in Weitzman (1973).[4]

3. SUFFICIENT CONDITIONS FOR THE OPTIMAL PATH

We start by establishing the sufficient conditions for a proposed path to be optimal. Because we do not rule out infinite momentary loss, we need to distinguish between paths with respect to whether their sum-of-payoff stream is bounded from below or not. Define the set of paths with sum of payoff stream bounded from below for a given state $\overline{x} \in X_0$ as follows:

$$G_{\overline{x}} = \left\{ \langle x_t, u_t\rangle_{t=0}^{\infty} \;\middle|\; \langle x_t, u_t\rangle_{t=0}^{\infty} \in F_{0,\overline{x}} \text{ and } \liminf_{T\to\infty} \sum_{t=0}^{T} u_t > -\infty \right\}.$$

THEOREM 1. *Given the initial state $\overline{x} \in X_0$, suppose that $\langle x_t, u_t\rangle_{t=0}^{\infty} \in G_{\overline{x}}$ and that there is a sequence $p \in (P^*)^{\mathbf{N}}$ such that for all $\langle y_t, v_t\rangle_{t=0}^{\infty} \in F_{0,y_0}$ with $y_0 \leq_V \overline{x}$ and $y_0 \in X_0$,*

$$\forall t \quad u_t + p_{t+1}(x_{t+1}) - p_t(x_t) \geq v_t + p_{t+1}(y_{t+1}) - p_t(y_t), \tag{2}$$

and for all $\langle y_t, v_t\rangle_{t=0}^{\infty} \in G_{y_0}$ with $y_0 \leq_V \overline{x}$ and $y_0 \in X_0$,

$$\limsup_{t\to\infty} p_t(x_t - y_t) \leq 0. \tag{3}$$

Then $\langle x_t, u_t\rangle_{t=0}^{\infty}$ is optimal.

Proof. It must be shown that $\liminf_{T\to\infty} \sum_{t=0}^{T}(u_t - v_t) \geq 0$ holds for any sequence $\langle y_t, v_t\rangle_{t=0}^{\infty} \in G_{y_0}$ satisfying $y_0 \leq_V \overline{x}$ and $y_0 \in X_0$ and

$\limsup_{t\to\infty} p_t(x_t - y_t) \leq 0$. If $\langle y_t, v_t\rangle_{t=0}^{\infty} \notin G_{y_0}$, then this conclusion holds trivially. Suppose $\langle y_t, v_t\rangle_{t=0}^{\infty} \in G_{y_0}$. The result follows from the following set of inequalities, holding for all feasible sequences with $y_0 \leq_V x_0 = \overline{x}$. The last line uses condition (3) together with the fact that $-\limsup(-a_t) = \liminf(a_t)$:

$$\liminf_{T\to\infty} \sum_{t=0}^{T} [u_t - v_t] = \liminf_{T\to\infty} \Bigg\{ [p_0(\overline{x}) - p_0(y_0)]$$

$$+ \left[\sum_{t=0}^{T} \{[u_t + p_{t+1}(x_{t+1}) - p_t(x_t)] - [v_t + p_{t+1}(y_{t+1}) - p_t(y_t)]\} \right]$$

$$- [p_{T+1}(x_{T+1}) - p_{T+1}(y_{T+1})] \Bigg\}$$

$$\geq \liminf_{T\to\infty} \left\{ 0 + \left[\sum_{t=0}^{T} 0 \right] - [p_{T+1}(x_{T+1}) - p_{T+1}(y_{T+1})] \right\}$$

$$\geq 0.$$

■

Condition (2) is about the existence of linear functionals valuing the state space such that the proposed optimal path solves a period-by-period optimization problem. Condition (3) is a general form of TVC describing the asymptotic behavior. Next we show that, under additional assumptions, it is equivalent to the familiar form $\lim_{t\to\infty} p_t(x_t) = 0$. We have stated Theorem 1 in terms of (3) to emphasize that the more familiar form is a special case. What really matters for the long-run behavior of the optimal sequence of state variables is that its value is asymptotically lower than under any other feasible path. For a discussion of various presentations of the TVC, see Michel (1990).

A.3. *There exists a sequence* $v = \langle v_t\rangle_{t=0}^{\infty}$ *such that* $[(\overline{x}, v_0), (0, v_1), (0, v_2)\ldots] \in G_{\overline{x}}$.

A.4. $[(x, r, y) \in \Phi_t, x \leq_V x', y' \leq_V y$ *and* $r' \leq r] \Longrightarrow (x', r', y') \in \Phi_t$.

A.5. *The state variable is restricted to the positive cone of* V*; i.e.,* $x_t \in P$ *for all* t.

(A.3) states that some finite payoff stream is feasible when we start from the given initial state, dispose of it immediately, and continue forever in state 0. Together with the free disposal assumption (A.4) and the non-negativity assumption (A.5), restricting the optimal path to be an element of $G_{\overline{x}}$ is justified by (A.3). The following proposition establishes the equivalence result.

PROPOSITION 1. *Suppose that the assumptions of Theorem 1 hold. Additionally assume (A.2)–(A.5). Then the TVC (3) is equivalent to* $\lim_{t\to\infty} p_t(x_t) = 0$.

Proof. To show necessity, use the path $\langle(\overline{x}, v_0), (0, v_1), (0, v_2)\ldots\rangle \in G_{\overline{x}}$ in condition (3). It reads as $\limsup_{t\to\infty} p_t(x_t) \leq 0$. By (A.4), we have $(x_t, u_t, 0) \in \Phi_t$ for all t. Applying condition (2),

$$\forall t \quad u_t + p_{t+1}(x_{t+1}) - p_t(x_t) \geq u_t + p_{t+1}(0) - p_t(x_t)$$
$$\implies \forall t \;\; p_{t+1}(x_{t+1}) \geq 0,$$

which implies that $0 \leq \liminf_{t\to\infty} p_t(x_t) \leq \limsup_{t\to\infty} p_t(x_t) \leq 0$; thus $\lim_{t\to\infty} p_t(x_t) = 0$.

The converse result holds under (A.5). Suppose $\lim_{t\to\infty} p(x_t) = 0$. Because $y_t \in P$ and $p_t \in P^*$, we have $p_t(y_t) \geq 0$ for all t. This implies that $\liminf_{t\to\infty} p_t(y_t) \geq 0$. The result follows from the following set of inequalities:

$$\limsup_{t\to\infty} p_t(x_t - y_t) \leq \limsup_{t\to\infty} p_t(x_t) + \limsup_{t\to\infty} p_t(-y_t) \leq -\liminf_{t\to\infty} p_t(y_t) \leq 0.$$

■

4. NECESSARY CONDITIONS FOR THE OPTIMAL PATH

In this section, we show that a converse result to Theorem 1 exists under some additional conditions: if a path is optimal, there exists a sequence of linear functionals (i.e., support prices) $p \in (P^*)^{\mathbf{N}}$ such that the optimal path solves the period-by-period optimization problem (2) and its asymptotic behavior satisfies the TVC (3).

The construction of support prices involves the separation of a feasible set from a set of paths yielding a higher sum of payoffs. We thus start by stating the well-known separating hyperplane theorem and proceed through several lemmata. In what follows, we refer to two convex sets A, B and make several assumptions about them. The relevant counterparts to these sets and assumptions in our environment will be introduced later.

THEOREM 2 (Separating Hyperplane Theorem). *Let* $A, B \subset S$ *be convex sets in a normed vector space* S. *Assume that* $\mathrm{int}(A) \cup \mathrm{int}(B) \neq \emptyset$ *and* $\mathrm{int}(A) \cap \mathrm{int}(B) = \emptyset$. *Then there is a continuous linear functional* q, *not identically equal to zero on* S, *such that for all* $x \in A$ *and all* $y \in B$, $q(x) \geq q(y)$.

Proof. The reader can refer to p. 133 in Luenberger (1969) for a proof. ■

As mentioned previously, the objects of interest are in the space $V \times \mathbf{R}$. The next lemma enables us to express a linear functional in this space as a sum of a functional defined over its subspace V and a real component.

LEMMA 1. *If $q : V \times \mathbf{R} \to \mathbf{R}$ is a continuous linear functional, then there exist a continuous linear functional $p : V \to \mathbf{R}$ and a real number r such that, if $z = (x, y) \in V \times \mathbf{R}$, then $q(z) = p(x) + ry$.*

Proof. Define $p(x) = q(x, 0)$ and $r = q(0, 1)$. Clearly, $p : V \to \mathbf{R}$ is a continuous functional because it is the projection of q, which is a continuous functional, over V. Its linearity is obvious. Also, $r \in \mathbf{R}$.

We check that $q(z)$ defined as $q(z) = q(x, 0) + q(0, 1)y$, where $z = (x, y) \in V \times \mathbf{R}$ is still a continuous linear functional. It is continuous because it is the sum of two continuous functions. It is a linear functional because for all $z_1, z_2 \in V$ and $\alpha, \beta \in \mathbf{R}$, the following holds:

$$\begin{aligned} q(\alpha z_1 + \beta z_2) &= p(\alpha x_1 + \beta x_2) + r(\alpha y_1 + \beta y_2) \\ &= q(\alpha x_1 + \beta x_2, 0) + q(0, 1)[\alpha y_1 + \beta y_2] \\ &= \alpha q(x_1, 0) + \alpha q(0, y_1) + \beta q(x_2, 0) + \beta q(0, y_2) \\ &= \alpha q(x_1, y_1) + \beta q(x_2, y_2) = \alpha q(z_1) + \beta q(z_2). \end{aligned}$$

■

The following lemma puts more structure on A and B to obtain a sharper description of the separating hyperplane.

LEMMA 2. *Let B be a convex set with $(x, s) \in \mathrm{cl}(B)$. Define $A = \{(x - x', s + s') \mid x' \in P$ and $s' \in \mathbf{R}_+\}$. Suppose A and B satisfy the conditions of Theorem 2. Moreover, suppose the following statements about B hold:*

$$\forall x' \in V \ \ \exists a > 0 \ \ \exists s' \in \mathbf{R} \quad (x + \alpha x', s') \in B, \tag{4}$$

$$\exists s' \quad s > s' \text{ and } (x, s') \in B, \tag{5}$$

$$\exists (x'', s'') \in B \quad s'' > s. \tag{6}$$

Then there exists a separating hyperplane $q(x, y) = -p(x) + ry$ such that $p \in P^*$, $p \neq 0$, and $r > 0$.

Proof. First show that $r > 0$. Note that $(x, s) \in A$. Take any $s'' > s$. By definition, $(x, s'') \in A$. Take $(x, s') \in B$ with $s' < s$, the existence of which is given by (5). We have $s'' > s > s'$. By Theorem 2, there exists a continuous linear functional $q \neq 0$ separating A and B, and by Lemma 1, we can represent q as $q(x, y) = -p(x) + ry$:

$$\begin{gathered} q(x, s'') \geq q(x, s'), \\ -p(x) + rs'' \geq -p(x) + rs', \\ rs'' \geq rs' \\ \Longrightarrow r \geq 0. \end{gathered}$$

Now suppose $r = 0$. Because $q \neq 0$, it must be that $p \neq 0$. So there is some $\tilde{x}$ such that $p(\tilde{x}) \neq 0$.

Define

$$x' = \begin{cases} \tilde{x} & \text{if} \quad p(\tilde{x}) < 0, \\ -\tilde{x} & \text{if} \quad p(\tilde{x}) > 0. \end{cases}$$

By definition, $p(x') < 0$. By condition (4), there exists $(x + \alpha x', s') \in B$ for some $\alpha > 0$. Because $(x, s) \in A$, by Theorem 2,

$$\begin{aligned} q(x, s) &\geq q(x + \alpha x', s'), \\ -p(x) + 0s &\geq -p(x) - \alpha p(x') + 0s', \\ \alpha p(x') &\geq 0, \end{aligned}$$

which is a contradiction. Hence $r > 0$.

Now show that $p \in P^*$. It is sufficient to show that for any $(x', s') \in A$ where $x' \neq x$ and $s' = s$, $q(x', s) \geq q(x, s)$, i.e., $p(x) \geq p(x')$.[5] Suppose to the contrary that there exists some $(x', s) \in A$ with $x' \neq x$ and $p(x) < p(x')$, i.e., $q(x', s) < q(x, s)$. Denote $q(x, s) - q(x', s) = \delta$. Because $(x, s) \in \text{cl}(B)$ and $(x, s) \in A$, there is a point $(\hat{x}, \hat{s}) \in B$ such that $\forall \delta > 0 \ |q(x, s) - q(\hat{x}, \hat{s})| < \delta$. By Theorem 2, $q(x, s) \geq q(\hat{x}, \hat{s})$ which implies that $q(x, s) - q(\hat{x}, \hat{s}) < \delta$. Thus

$$\begin{aligned} q(x, s) - q(\hat{x}, \hat{s}) < \delta &= q(x, s) - q(x', s), \\ q(x', s) &< q(\hat{x}, \hat{s}), \end{aligned}$$

which contradicts Theorem 2.

Finally, suppose that $\forall x \in V \ p(x) = 0$. Take $(x, s) \in A$ and $(x'', s'') \in B$ such that $s'' > s$. By Theorem 2,

$$\begin{aligned} q(x, s) = -p(x) + rs &\geq q(x'', s'') = -p(x'') + rs'', \\ rs &\geq rs'', \\ r &\leq 0, \end{aligned}$$

which contradicts the result $r > 0$ obtained previously. ■

We now define the sets A_t, B_t in our environment:

$$A_t = \{(x_t - \xi, \omega_t + v) \mid \xi \in P \text{ and } v \in \mathbf{R}_+\},$$

$$B_t = \Bigg\{(\xi, v) \in V \times \mathbf{R} \cup \{-\infty\} \mid \xi \in X_t \text{ and } v = \liminf_{T \to \infty} \sum_{\tau=t}^{T} v_\tau$$
$$\text{for some } \langle y_\tau, v_\tau \rangle_{\tau=t}^{\infty} \in F_{t,\xi} \Bigg\},$$

where $\langle x_t, u_t\rangle_{t=0}^{\infty}$ denotes the optimal path and ω_t the continuation value of the optimal path:

$$\omega_t = \liminf_{T\to\infty} \sum_{\tau=t}^{T} u_\tau.$$

A_t is the set of higher continuation values and lower levels of the state variable. B_t is the set of state and continuation value pairs where the latter can be attained by a feasible path starting from the former at date t. By construction, the point (x_t, ω_t) is both in A_t and in B_t for all t. The following assumption about Φ_t ensures that A_t and B_t are convex:

A.6. *For all t, Φ_t is convex.*

In order to apply Lemma 2, we make the following assumptions, which correspond to conditions (4)–(6):

A.7. *For all t,* $\forall x' \in V \;\; \exists a > 0 \;\; \exists s' \in \mathbf{R} \quad (x_t + \alpha x', s') \in B_t.$

A.8. *For all t,* $\exists(\psi, \sigma) \in B_t \;\; s.t. \;\; \sigma > \omega_t.$

A.9. $\text{int}(P) \neq \emptyset.$

(A.7), the counterpart of condition (4), asserts that the state vector along the optimal path is not on the boundary of the projection of B_t on its subspace V. (A.8), corresponding to (6), is an intuitive assumption in a growth model. One can always find a higher level of capital stock such that a path starting from this level yields a strictly higher payoff. An analogue of (5) is implied by (A.4). For any $(\xi, v) \in B_t$, there exists $v' < v$ such that $(\xi, v') \in B_t$ because one can freely dispose payoffs.

The last assumption (A.9) ensures that one of these sets A_t and B_t has a nonempty interior. Note that A_t is open when V is the two common state spaces we encounter: the finite-dimensional Euclidean space and $L^\infty(\Omega, Z, \pi)$ space.

We can now apply the separating hyperplane theorem to show that at each t, there is a hyperplane supporting B_t at $(x_t, \omega_t) \in V \times \mathbf{R}$, i.e., at the state variable and continuation value of the optimal path. For expositional purposes, Figure 1 illustrates a hyperplane separating the sets A_t and B_t.

LEMMA 3. *Suppose $\langle x_t, u_t\rangle_{t=0}^{\infty} \in G_{\overline{x}}$ is the optimal path with $x_0 = \overline{x}$. Suppose assumptions (A.1)–(A.4) and (A.7)–(A.9) hold. Define ω_t and B_t as before. Then, for each t, there exists some $p_t \in P^*$ with $p_t \neq 0$ satisfying*

$$-p_t(x_t) + \omega_t = \max\{-p_t(y_t) + \sigma \mid (y_t, \sigma) \in B_t\}. \qquad \textbf{(7)}$$

Proof. Note that, for B_t to be well defined for all t, the set $F_{t,\xi}$ should be nonempty for some $\xi \in X_t$. This is ensured by (A.1). Thus, $B_t \neq \emptyset$.

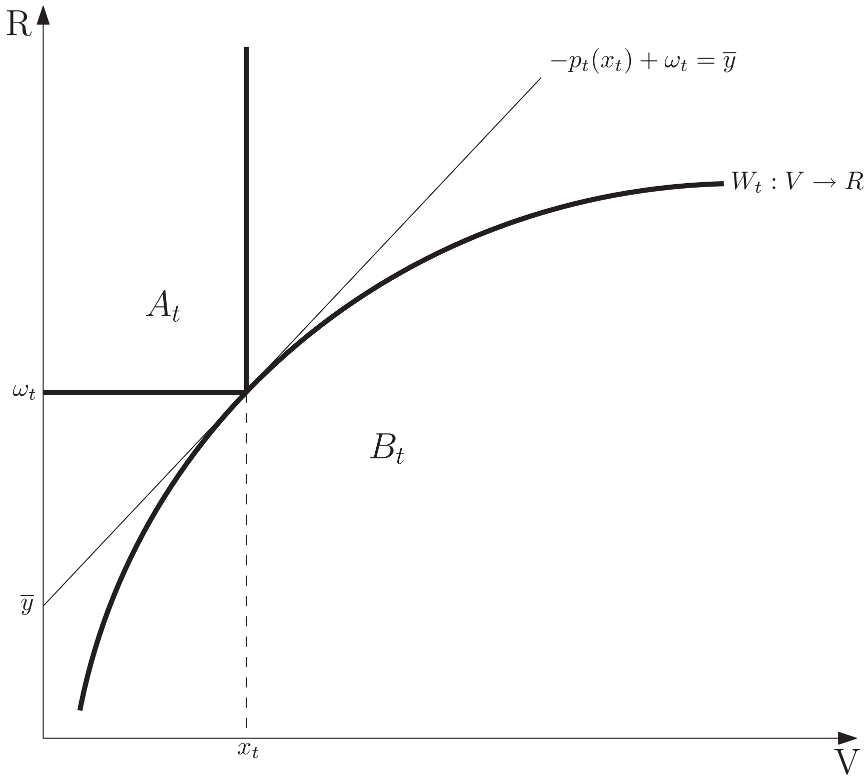

FIGURE 1. The separating hyperplane supports the set B_t at the optimal point.

First show that B_t is convex. Suppose it contains two elements (x_t^1, ω^1) and (x_t^2, ω^2) with $-\infty < \omega^1$ and $-\infty < \omega^2$. Define $(x_t^\alpha, \omega^\alpha) = [\alpha x_t^1 + (1 - \alpha)x_t^2, \alpha\omega^1 + (1 - \alpha)\omega^2]$ for some $0 < \alpha < 1$. By definition of B_t, we need to show the existence of a feasible path $\langle \tilde{x}_\tau, \tilde{u}_\tau \rangle_{\tau=t}^{\infty} \in F_{t,x_t^\alpha}$ satisfying $\tilde{x}_t = x_t^\alpha$ and $\omega^\alpha = \liminf_{T\to\infty} \sum_{\tau=t}^{T} \tilde{u}_\tau$.

For the two chosen elements in B_t, there are associated feasible paths $\langle x_\tau^1, u_\tau^1 \rangle_{\tau=t}^{\infty} \in F_{t,x_t^1}$ and $\langle x_\tau^2, u_\tau^2 \rangle_{\tau=t}^{\infty} \in F_{t,x_t^2}$ implying that $(x_\tau^1, u_\tau^1, x_{\tau+1}^1)$ and $(x_\tau^2, u_\tau^2, x_{\tau+1}^2)$ are in Φ_τ for all $\tau \geq t$. Take the path $\langle x_\tau^\alpha, u_\tau^\alpha \rangle_{\tau=t}^{\infty}$ defined by

$$x_\tau^\alpha = \alpha x_\tau^1 + (1 - \alpha)x_\tau^2,$$

$$u_\tau^\alpha = \alpha u_\tau^1 + (1 - \alpha)u_\tau^2.$$

By convexity of X_t and Φ_t for all $\tau \geq t$, we have $\langle x_\tau^\alpha, u_\tau^\alpha \rangle_{\tau=t}^{\infty} \in F_{t,x_t^\alpha}$.[6] Note that this is not necessarily the feasible sequence associated with $(x_t^\alpha, \omega^\alpha)$ because of the concavity of the liminf operator:

$$\overline{\omega}^{\alpha} = \liminf_{T\to\infty} \sum_{\tau=t}^{T} u_{\tau}^{\alpha} \;\geq\; \alpha\left(\liminf_{T\to\infty} \sum_{\tau=t}^{T} u_{\tau}^{1}\right) + (1-\alpha)\left(\liminf_{T\to\infty} \sum_{\tau=t}^{T} u_{\tau}^{2}\right) = \omega^{\alpha}.$$

By (A.2), the preceding expressions for ω^{α} and $\overline{\omega}^{\alpha}$ are well defined. If this holds with equality, $\langle x_{\tau}^{\alpha}, u_{\tau}^{\alpha}\rangle_{\tau=t}^{\infty}$ is the feasible path associated with $(x_t^{\alpha}, \omega^{\alpha})$ and we are done. Otherwise, define $d = \overline{\omega}^{\alpha} - \omega^{\alpha}$.

Let $\tilde{x}_{\tau} = x_{\tau}^{\alpha}$ for all $\tau \geq t$ and

$$\tilde{u}_{\tau} = \begin{cases} u_{\tau}^{\alpha} - d & \text{for } \tau = t, \\ u_{\tau}^{\alpha} & \text{for } \tau > t. \end{cases}$$

By (A.4), $\{\tilde{x}_{\tau}, \tilde{u}_{\tau}\} \in F_{t,x_t^{\alpha}}$ and it satisfies $\omega^{\alpha} = \liminf_{T\to\infty} \sum_{\tau=t}^{T} \tilde{u}_{\tau}$. This proves the convexity of the set B_t.

Define A_t as previously. This set is nonempty because $(x_t, \omega_t) \in A_t$. It is convex. (A.9) implies that $\text{int}(A_t) \neq \emptyset$. Hence $\text{int}(A_t) \cup \text{int}(B_t) \neq \emptyset$.

It can be proven by contradiction that $\text{int}(A_t) \cap \text{int}(B_t) = \emptyset$. Start by noting that by definition, $\omega_t = \sup\{v \mid (x_t, v) \in B_t\}$. Suppose, to the contrary, that $(y, v) \in \text{int}(A_t) \cap \text{int}(B_t)$. This assumption implies that $y \leq x_t$ and $\omega_t < v$. Because $(y, v) \in B_t$, by (A.4) we have $(x_t, v) \in B_t$, contradicting the supremum property of ω_t.

The sets A_t and B_t satisfy the condition of Theorem 2. There exists a continuous linear functional $q_t(\cdot, \cdot)$ such that

$$\forall (y_t, v) \in B_t \quad q_t(x_t, \omega_t) \geq q_t(y_t, v).$$

By Lemma 2, there exists a separating hyperplane with the representation $q_t(x, u) = -p_t(x) + r_t u$ where $r_t > 0$, $p_t \in P^*$ and $p_t \neq 0$. For all t, we can normalize prices by r_t without loss of generality:

$$\forall (y_t, \sigma) \in B_t \quad -p_t(x_t) + \omega_t \geq -p_t(y_t) + \sigma. \qquad \blacksquare$$

The final assumption to prove the necessary conditions for the optimal path is the differentiability of the feasibility set B_t at the optimal pair of a state and a continuation value. To be formal, define the value function for a given state at time t,

$$W_t(y_t) = \sup\{\sigma \mid (y_t, \sigma) \in B_t\}.$$

The time dependence of the value function is due to the potential nonstationarity of technology. Being the surface of the feasibility set B_t, the function $W_t : V \to \mathbf{R}$ is a transformation from a normed linear space V to $\mathbf{R}$. It gives the supremum sum of payoff streams feasible from a particular state. By (A.3), it is well defined for all feasible paths. By the principle of optimality, we have $W_t(x_t) = \omega_t$. By convexity of B_t, the function W_t is concave.

The generalization of differentiability of functions in normed vector spaces is given by Fréchet differentiability. We will assume that for all t, W_t is Fréchet differentiable at x_t relative to the projection of B_t to its first component. This derivative coincides with the derivative of the supporting hyperplane with respect to x_t. The linear functional $p_t : V \to \mathbf{R}$ has a derivative at x_t denoted by $\nabla p_t(x_t)$.

Define the projection of B_t on V:

$$\tilde{B}_t = \left\{ \xi \ \mid \ \exists v \ \text{ s.t. } \ (\xi, v) \in B_t \right\}.$$

A.10. *For all t, $W_t(.)$ is Fréchet differentiable at x_t relative to $\tilde{B}_t$ and $\nabla p_t(x_t)$ is the Fréchet differential of $W_t(.)$ at this point. In other words, for all $y \in \tilde{B}_t$,*

$$\lim_{\delta \to 0^+} \frac{\left| W_t[x_t + \delta(y - x_t)] - W_t(x_t) - \delta \ \nabla p_t(x_t) \cdot (y - x_t) \right|}{\delta \ \|y - x_t\|} = 0.$$

Differentiability establishes the link between the price functional and the value function. In Lemma 3, after showing the existence of a price functional $(-p_t, r_t)$ supporting the feasibility set at the optimal point, we normalized the price coefficient of current utility r_t to unity. The price functional $p_t : V \to \mathbf{R}$ is in terms of period-t utility. With differentiability, the price functional has an informational role about the variation of the value of the state around its optimal level.

We finish the section by establishing the necessary conditions for optimality:

THEOREM 3. *Suppose $\langle x_t, u_t \rangle_{t=0}^{\infty} \in G_{\overline{x}}$ is the optimal path for $x_0 = \overline{x}$ and the assumptions (A.1)–(A.10) are satisfied. Then there exists a sequence of $p \in (P^*)^{\mathbf{N}}$ such that for all $\langle y_t, v_t \rangle_{t=0}^{\infty} \in F_{0,y_0}$ with $y_0 \ \leq_V \ \overline{x}$ and $y_0 \in X_0$,*

$$\forall t, \quad u_t + p_{t+1}(x_{t+1}) - p_t(x_t) \geq v_t + p_{t+1}(y_{t+1}) - p_t(y_t), \tag{8}$$

and for all $\langle y_t, v_t \rangle_{t=0}^{\infty} \in G_{y_0}$ with $y_0 \ \leq \ \overline{x}$ and $y_0 \in X_0$,

$$\limsup_{t \to \infty} p_t(x_t - y_t) \leq 0. \tag{9}$$

Proof. Define A_t, B_t, and ω_t as before. By Lemma 3, for all t, there exists $p_t \in P^*$ with $p_t \neq 0$ satisfying

$$-p_t(x_t) + \omega_t = \max\{-p_t(\xi) + v \mid (\xi, v) \in B_t\}. \tag{10}$$

For all feasible paths and for all $t \geq 0$, we have $(y_t, \liminf_{T \to \infty} \sum_{\tau=t}^{T} v_\tau) \in B_t$ by definition of B_t. Using this information and the value function W_t as defined earlier, (10) reads as

$$\forall t \quad \forall y_t \in \tilde{B}_t \quad W_t(x_t) - p_t(x_t) \geq W_t(y_t) - p_t(y_t). \tag{11}$$

W_t satisfies the principle of optimality. For the optimal path and for any other feasible path the following conditions hold:

$$W_t(x_t) = u_t + W_{t+1}(x_{t+1}),$$

$$\forall(v_t, y_{t+1}) \quad [(y_t, v_t, y_{t+1}) \in \Phi_t \Longrightarrow W_t(y_t) \geq v_t + W_{t+1}(y_{t+1})].$$

Substituting these into (11), we obtain

$$\forall t \quad \forall(y_t, v_t, y_{t+1}) \in \Phi_t \quad u_t + W_{t+1}(x_{t+1}) - p_t(x_t) \geq v_t + W_{t+1}(y_{t+1}) - p_t(y_t). \tag{12}$$

We now show that the sequence of $\{p_t\} \in (P^*)^{\mathbf{N}}$ constructed by the price functionals satisfying (10) for all t also satisfies (8) for all t. By way of contradiction, suppose at some t, there exists a feasible (y_t, u_t, y_{t+1}) such that

$$u_t + p_{t+1}(x_{t+1}) - p_t(x_t) < v_t + p_{t+1}(y_{t+1}) - p_t(y_t).$$

Let $y_t^\varepsilon = \varepsilon y_t + (1-\varepsilon)x_t$ for $\varepsilon \in (0, 1)$. Define v_t^ε and y_{t+1}^ε in a similar fashion. For ε small enough,

$$u_t + p_{t+1}(x_{t+1}) - p_t(x_t) < v_t^\varepsilon + p_{t+1}(y_{t+1}^\varepsilon) - p_t(y_t^\varepsilon). \tag{13}$$

By convexity of the technology set Φ_t, such a $(v_t^\varepsilon, y_t^\varepsilon, y_{t+1}^\varepsilon)$ is feasible. Using the differentiability of $W(\cdot)$ at x_{t+1} with $\nabla W(x_{t+1}) = \nabla p_{t+1}(x_{t+1})$, the following holds:

$$W_{t+1}(y_{t+1}^\varepsilon) = W_{t+1}(x_{t+1}) + \nabla p_{t+1}(x_{t+1}) \cdot (y_{t+1}^\varepsilon - x_{t+1}) + o(\varepsilon), \tag{14}$$

where $o(\varepsilon)$ is an asymptotically negligible term. By linearity of the price functional $p_{t+1}(.)$, we have $\nabla p_{t+1}(x_{t+1}) \cdot (y_{t+1}^\varepsilon - x_{t+1}) = p_{t+1}(y_{t+1}^\varepsilon - x_{t+1}) = p_{t+1}(y_{t+1}^\varepsilon) - p_{t+1}(x_{t+1})$. Then (14) reads as

$$p_{t+1}(y_{t+1}^\epsilon) = W_{t+1}(y_{t+1}^\epsilon) - W_{t+1}(x_{t+1}) + p_{t+1}(x_{t+1}) - o(\epsilon). \tag{15}$$

Substituting $p_{t+1}(y_{t+1}^\epsilon)$ from (15) into (13), we obtain

$$u_t + p_{t+1}(x_{t+1}) - p_t(x_t) < v_t^\epsilon + W_{t+1}(y_{t+1}^\epsilon) - W_{t+1}(x_{t+1}) + p_{t+1}(x_{t+1}) - o(\epsilon) - p_t(y_t^\epsilon).$$

For ε small enough,

$$u_t + W_{t+1}(x_{t+1}) - p_t(x_t) < v_t^\epsilon + W_{t+1}(y_{t+1}^\epsilon) - p_t(y_t^\epsilon),$$

which contradicts (12). Thus (8) holds for all t.

To prove the TVC (9), take any $\langle y_t, v_t\rangle_{t=0}^{\infty} \in G_{y_0}$ with $y_0 \leq_V \overline{x}$. Again, by definition, $(y_t, \liminf_{T\to\infty} \sum_{\tau=t}^{T} v_\tau) \in B_t$ for all t in any such path. By (10),

$$-p_t(x_t) + \liminf_{T\to\infty} \sum_{\tau=t}^{T} u_\tau \geq -p_t(y_t) + \liminf_{T\to\infty} \sum_{\tau=t}^{T} v_\tau,$$

$$\implies \liminf_{T\to\infty} \sum_{\tau=t}^{T} u_\tau - \liminf_{T\to\infty} \sum_{\tau=t}^{T} v_\tau \geq p_t(x_t - y_t).$$

Taking $\limsup_{t\to\infty}$, the left side is zero because both payoff sequences have convergent sums by assumption and we get $\limsup_{t\to\infty} p_t(x_t - y_t) \leq 0$. ■

Theorem 1 and Theorem 3 jointly characterize the necessary and sufficient conditions for dynamic optimization.

5. DISCUSSION

We now compare our proof with earlier work. Weitzman (1973) uses an inductive argument. Starting with the first period, the separation of preferred and feasible sets in one period leads to the separation of the same sets in the following period with the payoff maximization condition (8) as an outcome. We separate these sets period by period (Lemma 3). To derive the intratemporal payoff maximization condition (8) with current and subsequent periods' state variables, we need additional assumptions. Our differentiability assumption at the boundary of the feasibility set helps to link the marginal change in the continuation value of a particular level of the state variable to its shadow price, thus highlighting the economic intuition about the role of prices. This is in line with Kamihigashi (2001), who also assumes the differentiability of the value function.

Also note that under both the sufficient and necessary conditions, transversality was only established to compare the optimal path to feasible paths with a sum that is bounded from below (the set G_{y_0} with $y_0 \leq_V \overline{x}$). This is similar to Michel (1990), who proves the TVC only for domains with finite loss.

The nonempty-interior assumption (A.9) also seems to be indispensable. Weitzman (1973) does not need to assume it because he restricts the analysis to Euclidean space. In contrast, McKenzie (1974) also makes this assumption in deriving support prices for weakly maximal paths in an infinite-horizon optimal growth model. Kamihigashi (2003) directly assumes the existence of an optimal feasible path because he is only interested in necessary conditions for optimality.

6. THE NO-PONZI CONDITION IN FISCAL PROBLEMS

We finish by demonstrating a useful application of the characterization of optimal paths introduced earlier. Whereas transversality is an *optimality* condition in infinite-horizon optimal problems, an analogous mathematical condition, called the no-Ponzi condition, is widely used in economics as a *constraint* on the action space of agents. For example, in models of fiscal policy, the no-Ponzi condition appears as a constraint on government borrowing in the limit to prevent overaccumulation of debt:

$$\lim_{t\to\infty}\left(\prod_{s=0}^{t-1}\frac{1}{R_s}\right)b_t = 0,$$

where b_t is the outstanding bond supply and R_s is the interest rate at period t.

Here we derive the no-Ponzi constraint on government borrowing from the optimality conditions of households' savings problem. An outline of this idea was given by Sims (1994):

> Government debt b_t cannot be unbounded above, given the boundedness of the price level P_t, because of the following transversality argument: if real debt has non-zero probability of growing arbitrarily large in an equilibrium with fixed interest rate $\overline{R}$ and with P bounded away from zero and infinity, it must eventually get larger than the level $\overline{b}$ such that $(\overline{R} - 1)\overline{b}P_{min}/P_{max} > \overline{\tau}$. This level $\overline{b}$ is high enough that with certainty the interest rate on it exceed the fixed level of real taxation $\overline{\tau}$ forever. At such a point, it appears feasible to the agent for him to reduce his bond holdings back to $\overline{b}$ and thereafter to consume at or above some positive minimum level forever...Thus the original candidate equilibrium path cannot have represented a solution to the agent's maximization problem and cannot have been an equilibrium. (p. 387)

Note that the necessity of the TVC (that is, Theorem 3) is what will be relevant here. This is distinct from the typical applications of transversality in macroeconomics; for example, most of those in Stokey and Lucas (1989), in which the TVC is ensured by a model specification involving discounting of a bounded payoff function, and that fact is used to justify the characterization of an agent's optimal behavior in terms of shadow prices or a value function.

Consider an economy populated by a continuum of identical households and a government. Households are endowed with an initial bond holding of b_0 and a per-period consumption good normalized to unity. They have lifetime utility as a discounted sum of the instantaneous utility $u(c)$ where the function $u(\cdot)$ satisfies the usual assumptions.

Households take a sequence of taxes $\langle T_t\rangle_{t=0}^{\infty}$ and one-period-ahead interest rates $\langle R_t\rangle_{t=0}^{\infty}$ to be paid on bond holdings as given. The household problem is to determine the optimal demand for bonds that solves

$$\max_{\langle b_t\rangle_{t=1}^{\infty}} \sum_{t=0}^{\infty}\beta^t u(c_t),$$

subject to b_0 and

$$\forall t \quad c_t + \frac{b_{t+1}}{R_t} = 1 - T_t + b_t.$$

We reformulate the problem in the language of our framework. Let $(b_t, v_t, b_{t+1}) \in \Phi_t$ if and only if

$$v_t = \beta^t u\left(1 - T_t + b_t - \frac{b_{t+1}}{R_t}\right),$$

$$b_{t+1} \leq R_t(1 - T_t + b_t),$$

and $b_t \in \mathbf{R}_+$ for all t. With a suitable concavity assumption on $u(\cdot)$, the set Φ_t is convex.

By Theorem 1 and Theorem 3, a path $\langle b_t^*, v_t^*\rangle_{t=0}^{\infty}$ is a solution to this problem if and only if there exist a sequence of nonnegative prices $\langle q_t\rangle_{t=0}^{\infty}$ satisfying the following conditions:

(i) At $t = 0$,

$$v_0^* + q_1 b_1^* \geq v_t + q_t b_1 \quad \text{for all} \quad (b_0, v_0, b_1) \in \Phi_0.$$

(ii) At $t \geq 1$,

$$v_t^* + q_{t+1} b_{t+1}^* - q_t b_t^* \geq v_t + q_{t+1} b_{t+1} - q_t b_t \quad \text{for all} \quad (b_t, v_t, b_{t+1}) \in \Phi_t.$$

(iii) In the limit,

$$\lim_{t\to\infty} q_t b_t^* = 0.$$

b_t^* maximizes the right-hand-side term in (ii) for all t. The first-order conditions with respect to b_t and b_{t+1} evaluated at optimum bond holdings are given by

$$\beta^t u'\left(1 - T_t + b_t^* - \frac{b_{t+1}^*}{R_t}\right) = q_t,$$

$$\beta^t u'\left(1 - T_t + b_t^* - \frac{b_{t+1}^*}{R_t}\right) \frac{1}{R_t} = q_{t+1},$$

which in turn establish the relationship between shadow prices and the interest rate:

$$q_{t+1} = \frac{q_t}{R_t}.$$

Letting $q_0 = 1$ without loss of generality, we get

$$q_t = \prod_{s=0}^{t-1} \frac{1}{R_s}.$$

Using this in the TVC (iii), and imposing market clearance such that the sequence of bond supply $\langle B_t\}_{t=1}^{\infty}$ equals the sequence of bond demand $\{b_t^*\rangle_{t=1}^{\infty}$ for all t, the no-Ponzi-scheme condition follows:

$$\lim_{t\to\infty}\left(\prod_{s=0}^{t-1}\frac{1}{R_s}\right)B_t = 0.$$

NOTES

1. Typically there are constraints, such as non-negativity of capital stocks, on what the state can be. Those constraints can be represented via the sets Φ_t to be defined later. In a deterministic environment with finitely many capital goods, it is natural to think of $V = \mathbf{R}^n$ with the Euclidean norm. For a stochastic environment with only one type of capital good (but in which the capital stock will evolve in a state-contingent manner), it might be reasonable to define the state space V to be $L^\infty(\Omega, \mathcal{B}, \pi)$, the space of equivalence classes of $\mathcal{B}$-measurable functions (that is, of random variables) on a probability space $(\Omega, \mathcal{B}, \pi)$ that are bounded in the essential supremum norm.

2. These functionals have an economic interpretation as prices. If the state space is $L^\infty(\Omega, \mathcal{B}, \pi)$, then the intuitively appropriate space of prices in most applications is $L^1(\Omega, \mathcal{B}, \pi)$, the space of functions that are integrable with respect to π. However, the normed-dual space of $L^\infty(\Omega, \mathcal{B}, \pi)$ is the larger space of finite measures. Assumptions can be placed on economic models that will guarantee that the relevant dual vectors correspond to elements of $L^1(\Omega, \mathcal{B}, \pi)$; see Bewley (1972), Mas-Colell and Richard (1991) and Aliprantis and Burkinshaw (2003). We conjecture that such assumptions can be adapted to the context of the dynamic model studied here, but that topic is beyond the scope of this paper.

3. In applications, this is proved either by showing recursively that a feasible path of finite length can be extended, or else by exhibiting an explicit infinite path that is feasible (for instance, completely to depreciate the capital stock immediately and then to leave it at 0 forever, if it is assumed possible to survive with 0 consumption).

4. In this regard, note also that in a model of discounted-payoff optimization, u_t can be interpreted as being the payoff discounted to date 0. If there is a bounded function $U\colon V \times V \to \mathbf{R}$ that satisfies $(x_t, u_t, x_{t+1}) \in \Phi_t \iff -\delta^t U(x_t, x_{t+1}) \le u_t \le \delta^t U(x_t, x_{t+1})$ for some $\delta \in (0, 1)$, then the partial sums in (1) converge.

5. Note that $A = \{(x', s') \mid x' \le x, s' \ge s\}$. Hence for all such (x', s), we have $(x - x') \in P$. Our claim is that $p(x) \ge 0$ for all $x \in P$. Hence we need to show that $p(x - x') \ge 0$ for all $(x', s') \in A$, where $x' \ne x$ and $s' = s$.

6. X_t is convex for all t because it is the projection of a convex set, Φ_t, on a subset.

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